STABILITY OF A BOUNDARY LAYER WITH TRANSONIC EXTERNAL-FLOW VELOCITIES

O. S. Ryzhov and I. V. Savenkov UDC 532.5

The notion of the free interaction of a boundary layer [1-3] has proven to be very useful in different areas of the mechanics of liquids and gases [4, 5], including the theory of hydrodynamic stability [6, 7]. It describes the structure of Tollmin–Schlichting waves at large Reynolds numbers and makes it possible to solve a number of important new problems concerning the reaction of the viscous sublayer to three-dimensional disturbances generated by a source localized in space [8, 9].

However, as is known, the theory of free interaction in its classical form predicts that forward-traveling waves propagating in the direction of the incident flow will be stable under the condition that the velocity of the flow be supersonic. On the other hand, the equations proposed so far for the transonic velocity range have proven unsuited for the solution of problems concerning the stability of viscous flows, despite the fact that these equations have been correctly used to predict separation [10]. It is obvious that there is a need for additional analysis of the initial system of Navier–Stokes equations in the indicated range, the goal here being to keep those terms which determine the loss of stability by the boundary layer. Such an analysis is made below.

1. Equations of Motion. We will examine the uniform flow of a compressible gas with the velocity $U_\infty{}^*$ about a flat plate. This velocity differs little from the velocity $a_\infty{}^*$ of sound waves in the medium. It is assumed that a local irregularity (roughness) exists at the distance L^* from the leading edge of the plate, flow in the neighborhood of this irregularity corresponding to the regime of free interaction [1-3]. Let $\nu_\infty{}^*$ be the kinematic viscosity of the gas. We introduce the Reynolds number $R = U_\infty{}^* L^*/\nu_\infty{}^* \to \infty$ and we express the small parameter $\varepsilon = R^{-1/8}$ in terms of it. We set $\delta = (M_\infty{}^2 - 1)/K_\infty{}'$, $K_\infty{}' = \text{const}$ and henceforth consider δ to be a second small parameter, thus establishing the incident flow as being transonic and having a Mach number M_∞ close to 1. The problem consists of determining a relationship between ε and δ for which the resulting approximate equations will be capable of describing the stability of the boundary layer and the development of self-generating oscillations in it.

In the free interaction regime, the time t^* and the space coordinates x^*, y^* are normalized as follows [1-3]:

$$t^* = (L^*/U^*_\infty)\varepsilon^2\delta^{-1/4}\beta^{-1}t',$$
$$x^* = L^*(1 + \varepsilon^3\delta^{-3/8}x'),\ y^* = L^*\varepsilon^5\delta^{-1/8}y', \tag{1.1}$$

where the additional parameter β is determined by the frequency scale and the components u^*, v^* and pressure p^* are expanded into asymptotic series

$$u^* = U_\infty{}^*[\varepsilon\delta^{-1/8}u'(t',\ x',\ y') + \ldots],$$
$$v^* = U_\infty^*[\varepsilon^3\delta^{1/8}v'(t', x', y') + \ldots],\ \ p^* = p_\infty^* + \rho_\infty^* U^{*2}_\infty[\varepsilon^2\delta^{-1/4}p'(t', x', y') + \ldots]. \tag{1.2}$$

Since the pressure does not change across the boundary layer, the ratio $T_w{}^*/T_\infty{}^*$ of wall temperature $T_w{}^*$ to the temperature of the incident flow $T_\infty{}^*$ is inversely proportional to the ratio $\rho_w{}^*/\rho_\infty{}^*$ of the analogous values of density ρ^*. With allowance for this fact, the equations for the lower wall region of the flow take the form

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0,\ \ \frac{\partial p'}{\partial y'} = 0,$$
$$\left(\frac{T_w^*}{T_\infty^*}\right)^{-1}\left(\beta\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'}\right) = -\frac{\partial p'}{\partial x'} + C\frac{T_w^*}{T_\infty^*}\frac{\partial^2 u'}{\partial y'^2}. \tag{1.3}$$

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 65-71, March-April, 1990. Original article submitted June 8, 1989.

In contrast to the classical Prandtl equations, the pressure gradient here is unknown beforehand and is calculated together with the velocity field. For simplicity, the plate is assumed to be thermally insulated. The letter C denotes the constant in the Chapman law $\lambda_w^*/\lambda_\infty^* = CT_w^*/T_\infty^*$, which connects the first viscosity coefficient $\lambda^* = \nu^*\rho^*$ with temperature.

In the external region of free interaction, the effects of viscosity and heat conduction are negligible in a first approximation. Thus, the flow in this region is nonvortical. We normalize the transverse coordinate in the region by means of $y^* = L^*\varepsilon^3\delta^{-7/8}y'_1$, while for the potential φ^* we write the expansion

$$\varphi^* = U_\infty^* L^*[x' + \varepsilon^5\delta^{-5/8}\varphi_1'(t', x', y') + \ldots]. \tag{1.4}$$

Having restricted the subsequent analysis to the inequality $\varepsilon\delta^{-1/8} \ll 1$ and having discarded small terms in the initial Navier–Stokes equations, we obtain

$$2\beta\frac{\partial^2\varphi_1'}{\partial t'\partial x'} + \varepsilon^{-1}\delta^{1/8}\left(\delta K'_\infty + 2\varepsilon^2\delta^{-1/4}\frac{\partial\varphi_1'}{\partial x'}\right)\frac{\partial^2\varphi_1'}{\partial x'^2} + \varepsilon^{-1}\delta^{9/8}\frac{\partial^2\varphi_1'}{\partial y_1'^2} = 0. \tag{1.5}$$

An attempt here to retain all of the components leads to the estimates $\delta \sim \varepsilon^{8/5}$ and $\beta \sim \varepsilon^{4/5}$ but leads to disappearance of the term with $\partial u'/\partial t'$ in the left side of the last equation of (1.3). The asymptotic theory of transient transonic flows formulated as a result correctly predicts separation of the boundary layer but is unsuitable for studying its stability, since the leading role in the development of wave processes in this case is played by the external region — where the velocity field is a potential field [10].

An alternative approach is based on the equality $\beta = 1$. It follows from this that $\delta \sim \varepsilon^{8/9}$, which makes it necessary to ignore the nonlinear term $(\partial\varphi'_1/\partial x')(\partial^2\varphi'_1/\partial x'^2)$ in Eq. (1.5). Allowing for this term is the heart of the theory of the transient motion of a gas in the transonic velocity region if the effect of dissipative factors (viscosity and heat conduction) are assumed to be negligible. Having taken $\delta = \varepsilon^{8/9}$ for simplicity, in the limit $\varepsilon \to 0$ we write the linear equation

$$2\frac{\partial^2\varphi_1'}{\partial t'\partial x'} + K'_\infty\frac{\partial^2\varphi_1'}{\partial x'^2} - \frac{\partial^2\varphi_1'}{\partial y'^2} = 0 \tag{1.6}$$

with the parameter $K'_\infty = (M_\infty^2 - 1)/\varepsilon^{8/9}$.

Although the thickness of the boundary layer has a passive role in the theory of free interaction, it is used in the procedure of combining the solutions for the viscous sublayer and the potential region of the flow. Passage to the limit, at which $y'_1 \to 0$ and $y' \to \infty$, gives

$$\frac{\partial\varphi_1'(t', x', 0)}{\partial x'} = -p'(t', x'),\quad \frac{\partial\varphi_1'(t', x', 0)}{\partial y_1'} = -\frac{\partial A'(t', x')}{\partial x'},$$
$$u' - \lambda C^{-1/2}(T_w^*/T_\infty^*)y' \to \lambda C^{-1/2}(T_w^*/T_\infty^*)A' \text{ at } y' \to \infty. \tag{1.7}$$

Here, the function $A'(t', x')$ has the meaning of the instantaneous displacement of the streamlines in the intermediate flow region, while the constant $\lambda = 0.3321$ characterizes dimensionless friction in the Blausius solution for the undisturbed boundary layer.

Proceeding on the basis of the group properties of Eqs. (1.3), (1.6) and conditions (1.7), we exclude the constants C and λ in them along with the ratio T_w^*/T_∞^*. For this purpose, we perform the affine transformation

$$t' = 2^{-2/9}C^{2/9}\lambda^{-14/9}(T_w^*/T_\infty^*)^{8/9}t,\quad x' = 2^{-3/9}C^{1/3}\lambda^{-4/3}(T_w^*/T_\infty^*)^{4/3}x,$$
$$y' = 2^{-1/9}C^{11/18}\lambda^{-7/9}(T_w^*/T_\infty^*)^{13/9}y,\quad y_1' = 2^{-7/9}C^{5/18}\lambda^{-13/9}(T_w^*/T_\infty^*)^{10/9}y_1 \tag{1.8}$$

of the independent variables (1.1) and the transverse coordinate y'_1 for the potential part of the flow. As regards the sought functions in (1.2), (1.4) and the displacement thickness A' from (1.7), the transformations for them appear as follows:

$$u' = 2^{-1/9}C^{1/9}\lambda^{2/9}(T_w^*/T_\infty^*)^{4/9}u,\quad v' = 2^{1/9}C^{7/18}\lambda^{7/9}(T_w^*/T_\infty^*)^{5/9}v,$$

$$p' = 2^{-2/9}C^{2/9}\lambda^{4/9}(T_w{}^*/T_\infty{}^*)^{-1/9}p,$$
$$A' = 2^{-1/9}C^{11/18}\lambda^{-7/9}(T_w{}^*/T_\infty{}^*)^{13/9}A,\ \varphi'_1 = 2^{-5/9}C^{5/9}\lambda^{-8/9}(T_w{}^*/T_\infty{}^*)^{11/9}\varphi_1. \quad (1.9)$$

As a result, the system of equations

$$\partial u/\partial x + \partial v/\partial y = 0,\ \partial p/\partial y = 0,$$
$$\partial u/\partial t + u\partial u/\partial x + v\partial u/\partial y = -\partial p/\partial x + \partial^2 u/\partial y^2, \quad (1.10)$$

governing the flow in the viscous sublayer, along with the limiting conditions

$$u \to y,\ p \to 0 \text{ at } x \to -\infty,\ u - y \to A(t, x) \text{ at } y \to \infty \quad (1.11)$$

acquires a canonical form which is independent of the initial parameters of the problem. The similarity of the external velocity field, the potential φ_1 of which satisfies the equation

$$\partial^2\varphi_1/(\partial t\partial x) + K_\infty\partial^2\varphi_1/\partial x^2 - \partial^2\varphi_1/\partial y_1{}^2 = 0, \quad (1.12)$$

is determined by a single coefficient differing from ±1

$$K_\infty = 2^{-8/9}C^{-1/9}\lambda^{-2/9}(T_w{}^*/T_\infty{}^*)^{-4/9}(M_\infty{}^2 - 1)/\varepsilon^{8/9}, \quad (1.13)$$

since the boundary conditions for it are written in the form

$$\frac{\partial\varphi_1(t, x, 0)}{\partial x} = -p(t, x);\quad \frac{\partial\varphi_1(t, x, 0)}{\partial y_1} = -\frac{\partial A(t, x)}{\partial x}. \quad (1.14)$$

2. Eigenvalue Problem. We will begin with internal waves in the mechanical system being examined here, consisting of a viscous sublayer and a potential flow. With this in mind, we augment the limit relations (1.11) for system (1.10) with the conditions

$$u = v = 0 \text{ at } y = 0 \quad (2.1)$$

expressing adhesion of the gas to the surface of the plate. This leads us to a closed problem which also contains Eq. (1.12) together with boundary conditions (1.14). The boundary conditions contain the function A(t, x), which is unknown beforehand.

We take the solution and isolate the part $u = y$, $v = p = A = \varphi_1 = 0$, which describes the shear flow on the plate. The oscillations imposed on this flow will be proportional to the amplitude parameter a. Then, as usual, we put

$$(u - y,\ v,\ p,\ A,\ \varphi_1) = a(u_0,\ v_0,\ p_0,\ A_0,\ \varphi_0)\exp(\omega t + ikx). \quad (2.2)$$

Inserting (2.2) into the equation for the viscous sublayer and passing to the limit at $a \to 0$, we obtain a system of ordinary differential equations whose integration, with allowance for (2.1), yields the following [6, 7]

$$\Phi(\Omega) = (ik)^{1/3}p_0/A_0,$$
$$\Phi = \frac{d\mathrm{Ai}(\Omega)}{d\zeta}\left[\int_\Omega^\infty \mathrm{Ai}(\zeta)\,d\zeta\right]^{-1},\quad \Omega = \omega(ik)^{-2/3}$$

[$\mathrm{Ai}(\zeta)$ is an Airy function which decays exponentially in the sector $-\pi/3 < \arg\zeta < \pi/3$].

The ratio p_0/A_0 is established from the solution of problem (1.12), (1.14) for the upper potential region of the flow. It expresses the connection between excess pressure and the displacement thickness in the transonic velocity range. It is easily seen that $p_0/A_0 = k^2/\lambda(\omega, k)$; $\lambda = \sqrt{ik(\omega + ikK_\infty)}$, while $\mathrm{Re}\lambda \geq 0$ by virtue of the requirement of degeneration of the perturbations at $y_1 \to \infty$, x = const.

This range is characterized by the fact that terms with a time derivative enter both into system (1.10) (for the viscous sublayer) and into Eq. (1.12) (which governs the development of the external potential oscillations). The structure of the resulting waves is determined by the interaction of two essentially nonstationary fields, only one of which is vortical. Not only is pressure induced by an increase or decrease in the displacement thickness and in turn actively influences the change in the latter, but it is also the deciding factor in the transmission of signals in the external region. This circumstance is reflected in the fact that the condition of free interaction, as the right side of the below dispersion relation which follows from it

$$\Phi(\Omega) = Q(\omega, k),\ Q = -(ik)^{7/3}/\lambda(\omega, k) \tag{2.3}$$

depends explicitly on the frequency ω. No similar situation was encountered earlier in an asymptotic analysis based on a multilayered subdivision of the region of disturbed motion of a viscous liquid which included subsonic and supersonic boundary layers [6, 7].

3. Dispersion Relation. We begin with the observation that all of the solutions Ω of the dispersion relation for any $k \to 0$ can be fixed by means of $\Omega \to \Omega_n^{(0)}$ ($\Omega_n^{(0)}$ is the n-th root of the derivative $d\mathrm{Ai}(\Omega)/d\zeta$). Thus, the dispersion relation determines the theoretical set of dispersion curves $\omega_n = (ik)^{2/3}\Omega_n$. Here, in the limit of small wave numbers, these curves behave as the analogous curves for subsonic and supersonic boundary layers [6, 7].

Now let k in the right side of (2.3) take real values. Meanwhile, by virtue of symmetry, it is sufficient to limit this relation to the semi-axis $k > 0$. In order to find the frequency of neutral oscillation of a gas with an amplitude which is constant over time, we put $\omega = -i\omega_0$. Here ω_0 is a real number. Designating $D = \omega_0 - kK_\infty$, we immediately conclude from the equality $\arg Q(-i\omega_0, k) = \begin{cases} \pi/6 & \text{at } D > 0, \\ -\pi/3 & \text{at } D < 0 \end{cases}$ that the dispersion relation in this case permits a unique solution satisfying the system

$$\omega_0 k^{-2/3} = \omega_*' (k_*')^{-2/3},\quad k^{7/3}/\sqrt{k(\omega_0 - kK_\infty)} = (k_*')^{4/3}$$

($\omega_*' = 2.298$ and $k_*' = 1.0005$ are the frequency and wave number of the neutral oscillations of the boundary layer in an incompressible fluid [6, 7]). The asymptotes of the solution $\omega_0 = \omega_*(K_\infty)$, $k = k_*(K_\infty)$ of this system have the form

$$\begin{aligned} &\omega_* = \omega_*' |K_\infty|^{1/4},\quad k_* = k_*' |K_\infty|^{3/8} && \text{at } K_\infty \to -\infty, \\ &\omega_* = k_* K_\infty,\quad k_* = k_*' (\omega_*'/(k_*' K_\infty))^3 && \text{at } K_\infty \to \infty. \end{aligned} \tag{3.1}$$

The subsequent analysis of (2.3) is based on Newton's iteration method, which we use to calculate the roots $\Omega_n(k)$; as the initial approximation for them at $k \ll 1$, we chose the quantities $\Omega_n^{(0)}$. All of the roots Ω_n beginning with the second give stable oscillations: $\mathrm{Re}\,\omega_n(k) \le 0$ for any real k and $n \ge 2$. The first root Ω_1 generates the dispersion curve $\omega_1(k)$, which at $k = k_*$ intersects the negative imaginary semi-axis at the point $\omega_1 = -i\omega_*$ corresponding to neutral pulsations. The perturbations become unstable at $k > k_*$, when $\mathrm{Re}\,\omega_1(k) > 0$. The trajectory of this root in the complex plane ω is shown in Fig. 1 for $K_\infty = -1$ (1) and 1 (2).

Qualitatively speaking, the path of both curves reminds us of how they were obtained for a boundary layer in an incompressible fluid [6, 7]. However, there is an important difference between them. The increment $\mathrm{Re}\,\omega_1$, characterizing the intensification of Tollmin–Schlichting waves in an incompressible fluid, approaches the constant value $\sqrt{2}/2$ at $k \to \infty$ in accordance with the asymptote

$$\omega_1 = -ik^2 + (\sqrt{2}/2)(1 - i) + \ldots \tag{3.2}$$

The asymptote of the first root of dispersion relation (2.3), valid in the transonic velocity range, appears as follows:

$$\omega_1 = -i(k^{5/3} + (1/3)K_\infty k + (1/3)K_\infty^2 k^{1/3}) + (\sqrt{2}/3)(1 - i)k^{1/6} + \ldots \tag{3.3}$$

under the condition that $k \to \infty$, while the value of K_∞ is fixed. It follows from this that the increment $\mathrm{Re}\,\omega_1 \to \infty$ as $\sqrt{2}k^{1/6}/3$, although its increase at moderate k is slight and does not have a significant effect on the form of the curves in Fig. 1.

The limiting behavior of $\mathrm{Re}\,\omega_1$ dictated by the asymptotes (3.2) and (3.3) contradicts the well-known fact that as $k \to \infty$ there should be a transition to stable oscillations with

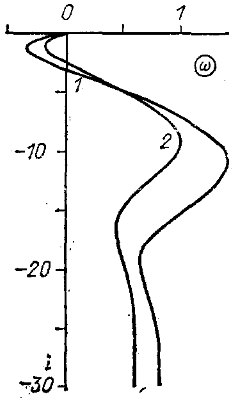

Fig. 1

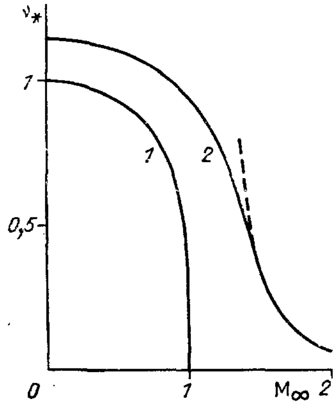

Fig. 2

an exponentially decaying amplitude, although this contradiction is expressed less clearly for an incompressible boundary layer. This transition is connected with the existence of a second pair of critical values of frequency and wave number, corresponding to the upper branch of the curve of neutral stability. An asymptotic analysis of the waves belonging to the neighborhood of this branch was made in [11, 12] for an incompressible fluid. It is based on a more complex perturbation structure which leads to renormalization of both the independent variables and the sought functions. The latter fact makes it possible to establish the scales of the above-mentioned second pair of critical values of frequency and wave number, calculate these quantities [13], and extend Eq. (3.2) into the neighborhood of the upper branch of the neutral curve by the method of combining asymptotic series. A similar analysis must be done for the transonic velocity range, but the development of unstable pulsations with characteristic times and lengths, given by Eqs. (1.1), will be determined by a distinct maximum of $\mathrm{Re}\,\omega_1$ at $k \sim 3$-4 (see Fig. 1).

In conclusion, let us evaluate the range of Mach numbers in which the above theory is valid. Let ν^* be the frequency in the initial system of units of measurement and let ν be the dimensionless frequency normalized by means of the expression

$$\nu = \left(\nu^* L^*/U_\infty^*\right) \varepsilon^2 C^{1/4} \lambda^{-3/2} \left(T_w^*/T_\infty^*\right) \omega_*'^{-1}.$$

The latter is convenient because its value ν_* for neutral oscillations propagating in a subsonic boundary layer with $M_\infty < 1$ will be [6, 7]

$$\nu_* = \left(1 - M_\infty^2\right)^{1/4}. \qquad (3.4)$$

As a simple check shows, by virtue of the transformation (1.8) for time, this value coincides with the limit ω_* which is approached by the above-examined frequency ω_0 at $K_\infty \to -\infty$ and which is predicted by the first formula of (3.1). It is evident from the definition (1.13) that $K_\infty \to -\infty$ when the difference $1 - M_\infty^2$ is positive and fixed, while the parameter $\varepsilon \to 0$. Thus, the critical frequency of oscillations propagating in a transonic flow matches the critical frequency calculated for subsonic flows as K_∞ decreases. Recalling the affine transformations (1.9), we can see that a similar situation holds in regard to excess pressure and the components of the vector of perturbed velocity. In other words, the mixed derivative $\partial^2\varphi_1/\partial t\partial x$ in Eq. (1.12) becomes negligibly small if $K_\infty \to -\infty$.

As regards the rate at which the frequency ν_* approaches its limiting values, it is evaluated as follows. In the transonic velocity range

$$\nu_* = \Delta\omega_*(K_\infty), \quad K_\infty = \left(M_\infty^2 - 1\right)/\Delta^4, \qquad (3.5)$$
$$\Delta = [4\varepsilon^2 C^{1/4} \lambda^{1/2} (T_w^*/T_\infty^*)]^{1/9}.$$

The parameter ε changes little with the variation of R, and we can assume that $\varepsilon \approx 0.2$ in the range of Reynolds numbers for the laminar-turbulent transition of interest to us here. The ratio T_w^*/T_∞^* for a thermally insulated plate obeys Crocco's law [14]

$$T_w^*/T_\infty^* = 1 + [(\varkappa - 1)/2] M_\infty^2,$$

i.e., $T_w^*/T_\infty^* \to (\varkappa + 1)/2$ at $M_\infty \to 1$ ($\varkappa$ is Poisson's adiabatic component). We will assume that the plate is in a flow of air with $T_\infty^* = 293$ K. Then $\varkappa = 1.4$, while $C \approx 0.79$; as a result, we obtain $\Delta \approx 0.778$. The Mach number M_∞ takes values from 0 to 2 with a change in the similarity parameter within the interval $-2.73 \le K_\infty \le 8.19$, from which it is evident that attainment of the limit $K_\infty \to -\infty$ is nearly impossible under practical conditions.

The curves representing Eqs. (3.4) and (3.5) are denoted by the numbers 1 and 2 in Fig. 2. Although they approach one another with a decrease in M_∞, there remains a 14% difference between them at $M_\infty = 0$ due to the finite value of ε. The critical frequency of oscillation in the transonic velocity range approaches its asymptotic value slowly. Conversely, the approach of the asymptote $\nu_* = \Delta^9(\omega_*'/k_*')^2/(M_\infty^2 - 1)$, established by the second formula of (3.1) for a supersonic boundary layer and shown by the dashed line in Fig. 2, is very rapid. At just $M_\infty = 1.5$, curve 2 nearly merges with this line.

The most important result of the theory that has been developed is the passage of the critical frequency (as the remaining relations) through the threshold value $M_\infty = 1$. Thus, its range of application covers a broad interval of subsonic and moderately supersonic velocities. An additional source of error is introduced into the theory when the frequency ω_*' of neutral oscillations of a boundary layer in an incompressible fluid is calculated from the limiting value, when $R \to \infty$. This source should be kept in mind when comparing the predictions of the asymptotic theory with data from wind-tunnel measurements [15].

LITERATURE CITED

1. V. Ya. Neiland, "Toward a theory of a laminar boundary layer in a supersonic flow," Izv. Akad. Nauk SSSR Mekh. Zhidk. Gaza, No. 4 (1969).
2. K. Stewartson and P. G. Williams, "Self-induced separation," Proc. R. Soc. London Ser. A, 312, No. 1509 (1969).
3. A. F. Messiter, "Boundary-layer flow near the trailing edge of a flat plate," SIAM Soc. Ind. Appl. Math J. Appl. Math, 18, No. 1 (1970).
4. F. T. Smith, "Steady and unsteady boundary-layer separation," Ann. Rev. Fluid Mech., 18 (1986).
5. V. V. Sychev, A. I. Ryban, Vik. V. Sychev, and G. L. Korolev, Asymptotic Theory of Separated Flows [in Russian], Nauka, Moscow (1987).
6. O. S. Ryzhov and E. D. Terent'ev, "Transitional regime characterizing the startup of a vibrator with a subsonic boundary layer on a plate," Prikl. Mat. Mekh., 50, No. 6 (1986).
7. O. S. Ryzhov and E. D. Terent'ev, "Vortex spots in the boundary layer," Fluid Dyn. Trans., 13 (1987).
8. O. S. Ryzhov and I. V. Savenkov, "Asymptotic theory of a wave packet on a plate," Prikl. Mat. Mekh., 51, No. 5 (1987).
9. O. S. Ryzhov and I. V. Savenkov, "Three-dimensional disturbances introduced by a harmonic oscillator into a boundary layer on a plate," Zh. Vychisl. Mat. Mat. Fiz., 28, No. 4 (1988).
10. O. S. Ryzhov, "Asymptotic methods in transonic flow theory," Proc. IUTAM Symp. Transsonicum III, Gottingen, 1988. Springer, Berlin et al. (1989).
11. V. I. Zhuk and O. S. Ryzhov, "Asymptotic solutions of the Orr—Sommerfeld equation yielding unstable oscillations at large Reynolds numbers," Dokl. Akad. Nauk SSSR, 268, No. 6 (1983).
12. V. I. Zhuk, "Asymptotic solutions of the Orr—Sommerfeld equation in regions adjacent to two branches of the neutral curve," Izv. Akad. Nauk SSSR Mekh. Zhidk. Gaza, No. 4 (1984).
13. R. J. Bodonyi and F. T. Smith, "The upper branch stability of the Blausius boundary layer, including nonparallel flow effect," Proc. R. Soc. London Ser. A, 375, No. 1760 (1981).
14. K. Stewartson, The Theory of Laminar Boundary Layers in Compressible Fluids, Clarendon Press, Oxford (1964).
15. V. V. Kozlov and O. S. Ryzhov, "Susceptibility of a boundary layer: asymptotic theory and experiment," in: Reports on Applied Mathematics [in Russian], Computer Center, Academy of Sciences of the USSR, Moscow (1988).